J. Appl. Maths Mechs Vol. 57, No. 2, pp. 221–236, 1993
Printed in Great Britain.

0021–8928/93 $24.00+.00

ANALYTICAL METHODS IN THE THEORY OF VIBRO-IMPACT SYSTEMS†

A. P. IVANOV

Moscow

(*Received* 11 *October* 1991)

A new approach to the study of vibro-impact systems is proposed, based on a continuous representation of motions with impacts in an auxiliary phase space. In the Lagrange-coordinate description, the phase trajectories experience discontinuities at the impact times. Direct investigation of such curves is difficult, as the standard topological concepts, such as neighbourhood and connectedness, break down. The traditional approach is to construct a point mapping at the level of the limiter [1]. With that choice of cross section, however, the Poincaré map is not continuous everywhere, not to speak of the impossibility of explicitly constructing the map in practice [2, 3].

These difficulties can be overcome by using standard qualitative methods: the Poincaré–Bendixson theory, Lyapunov's second method, etc. Several general results are established, touching on the nature of the equilibrium positions and periodic motions with impacts.

1. THE METHOD OF CONTINUOUS REPRESENTATION

CONSIDER a mechanical system with one degree of freedom and a unilateral constraint

$$x^{\cdot\cdot}=f(t,x,x^{\cdot}),\quad x\geqslant 0 \tag{1.1}$$

where f is a continuously differentiable function, either independent of t or periodic in t with period T. The motion at $x=0$ is defined as follows:

$$x^{\cdot\cdot}=\max\{0,f\}\quad\text{for}\quad x^{\cdot}=0;\quad x^{\cdot+}=-\kappa x^{\cdot-}\quad\text{for}\quad x^{\cdot}<0 \tag{1.2}$$

where the plus and minus superscripts denote the pre- and post-impact velocities, and κ is Newton's coefficient of restitution, $\kappa\in(0, 1)$. By (1.2), the phase trajectories of system (1.1) are discontinuous on the half-line $x=0$, $x^{\cdot}<0$ (Fig. 1a).

We wish to define new phase variables s, υ that are continuous functions of time. Put

$$x=|s|,\quad x^{\cdot}=F(s,\upsilon)$$

where F is a discontinuous function such that, when the phase trajectory in the (s, υ) plane intersects the $s=0$ axis, the boundary conditions of the impact for the variable $x^{\cdot}$ are automatically satisfied. If attention is confined to the case in which $s^{\cdot}$ and υ have the same sign, the intersection in question may occur when the trajectory crosses from the fourth quadrant of the (s, υ)-plane into the third (Fig. 1b) or from the second into the first; at the same time, the product $s\upsilon$ will change sign from minus to plus. Consequently, the second condition in (1.2)

†*Prikl. Mat. Mekh.* Vol. 57, No. 2, pp. 5–21, 1993.

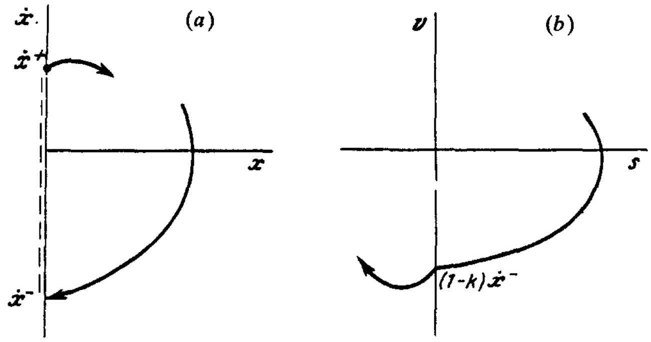

FIG. 1.

will be satisfied if

$$F(+0,\upsilon)=-\kappa F(-0,\upsilon) \quad \text{for} \quad \upsilon>0$$
$$F(-0,\upsilon)=-\kappa F(+0,\upsilon) \quad \text{for} \quad \upsilon<0$$

The simplest substitution satisfying these requirements is

$$x=|s|,\quad x^{\cdot}=R\upsilon \operatorname{sign} s,\quad R=1-k\operatorname{sign}(s\upsilon),\quad k=\frac{1-\kappa}{1+\kappa} \tag{1.3}$$

The variation of the new variables s, υ, by virtue of (1.3), (1.1), is described by the equations

$$s^{\cdot}=R\upsilon,\quad \upsilon^{\cdot}=R^{-1}\operatorname{sign} s f(t,|s|,R\upsilon\operatorname{sign} s) \tag{1.4}$$

in which the right-hand sides are discontinuous at $s=0$ or $\upsilon=0$. There is now a fairly sophisticated theory of systems with discontinuous right-hand sides (see [4]). The main merit of system (1.4) compared with (1.1) is that the phase trajectories in the (s, υ)-plane are continuous (see Fig. 1b). In absolutely elastic impacts, $\kappa=1$, $k=0$, and formulae (1.3) reduce to the substitution suggested in [5].

Equations (1.4) remain unchanged if the signs of s and υ are both inverted. The motion of the representative point in the (s, υ)-plane is uniquely defined if $s^2+\upsilon^2\neq 0$; uniqueness breaks down at the origin, for if

$$s=s_1(t),\quad \upsilon=\upsilon_1(t),\quad s_1(t_0)=\upsilon_1(t_0)=0 \tag{1.5}$$

is any solution of system (1.4), then the function

$$s=-s_1(t),\quad \upsilon=-\upsilon_1(t) \tag{1.6}$$

describes another solution. By (1.3), both solutions (1.5) and (1.6) correspond to the same trajectory in the $(x, x^{\cdot})$-plane.

Redefining the right-hand sides of system (1.4) on the lines of discontinuity away from the origin has no effect on the solution; by the first equation of (1.2), the correspondence (1.3) is preserved at $s=\upsilon=0$ if one puts

$$s^{\cdot}=0,\quad \upsilon^{\cdot}=(1-k)^{-1}\max\{0,f(t,0,0)\} \tag{1.7}$$

Then, if $f\leqslant 0$, the representative point remains at the origin ("grazing" motion) but if $f>0$, it leaves the origin along one of the curves (1.5) or (1.6).

The fact that the solutions of system (1.4) are not differentiable at $\upsilon=0$ is a consequence of the choice of the substitution (1.3). By using a function $F(s, \upsilon)$ of a more complicated form, one can construct an auxiliary system whose right-hand sides have discontinuities on the straight line $s=0$ only, for example

$$F(s,\upsilon)=\left(1-\frac{2k}{\pi}\operatorname{arctg}\frac{\upsilon}{s}\right)\upsilon \operatorname{sign} s \tag{1.8}$$

Example. Consider a linear oscillator with impacts against a limiter

$$x^{\cdot\cdot}+2bx^{\cdot}+a^2(x-x_0)=0, \quad x\geqslant 0 \tag{1.9}$$

where $a^2>b^2$ and x_0 describes the adjustment of the system: positive values correspond to clearance, and negative, to preload. We shall assume that $x_0=0$. The substitution (1.3) gives a system

$$s^{\cdot}=R\upsilon, \quad \upsilon^{\cdot}=-2b\upsilon-a^2R^{-1}s$$

which has the following general solution in each quadrant of the (s,υ)-plane

$$s=e^{-bt}(C_1^{(j)}\cos\delta t+C_2^{(j)}\sin\delta t), \quad \upsilon=R^{-1}s^{\cdot}, \quad \delta^2=a^2-b^2$$

The constants $C_{1,2}^{(j)}$ change whenever the phase trajectory intersects the coordinate axes, in such a way that the trajectory remains continuous. With initial conditions $s=0$, $\upsilon=\upsilon_0>0$ at $t=0$, we have $C_1^{(0)}=0$, $C_2^{(0)}=(1-k)\delta^{-1}\upsilon_0$ in the first and fourth quadrants. At $t=\pi/\delta$ the trajectory enters the third quadrant, and continuity demands that

$$C_1^{(1)}=0, \quad C_2^{(1)}=\sigma C_2^{(0)}, \quad \sigma=\kappa\exp(-\pi b/\delta)$$

Finally, for $t\in(2\pi/\delta, 3\pi/\delta)$, we have $C_1^{(2)}=0$, $C_2^{(2)}=\sigma^2C_2^{(0)}$, and so on.

The general appearance of the phase portrait depends on the constant σ. If $\sigma=1$, all the phase trajectories in the (s, υ)-plane are closed (Fig. 2a); if $\sigma<1$ they approach the origin asymptotically as $t\to+\infty$ (Fig. 2b); and if $\sigma>1$ they approach the origin asymptotically as $t\to-\infty$ (Fig. 2c).

2. CLASSIFICATION OF SINGULAR POINTS

We will study the singular points of system (1.1) in the autonomous case, i.e. when the right-hand side is not explicitly dependent on time. These are the roots of the equation $f(x^*, 0)=0$, if $x^*>0$, as well as the origin. In the first case the presence of a one-sided limiter does not affect the motion in a small neighbourhood of the singular point, so that the nature of the latter can be investigated by the usual methods.

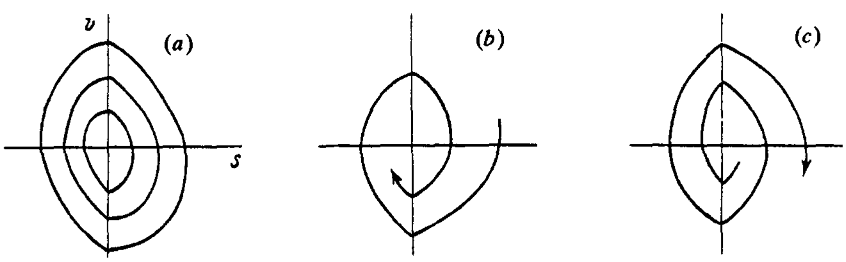

FIG. 2.

To analyse the neighbourhood of the point (0, 0), we use the representation (1.4). Here one can use the standard classification, as presented in [4], but the special form of system (1.4) will help to derive more complete results.

Following [4], we shall call a singular point of system (1.1) structurally stable if, for any function $g(x, x^{\cdot})$, sufficiently close to f in the metric of C_1, the system

$$x^{\cdot\cdot} = g(x, x^{\cdot}), \quad x \geqslant 0 \tag{2.1}$$

has a singular point of the same topological type.

Theorem 1. There exist exactly two types of structurally stable singular points of system (1.1) at the origin:

1. if $f_0 = f(0, 0) > 0$, all trajectories in some neighbourhood of the origin will leave that neighbourhood within a finite time (a "quasi-saddle-point" [6]);

2. if $f_0 < 0$, all trajectories in some neighbourhood of the origin will enter the singular point within a finite time (a "quasi-focus" or "sewn focus" [6]).

Singular points with $f_0 > 0$ do not maintain their topological type under variations of the function f, even variations that are small in the metric of C_r, $r \geqslant 1$.

Proof. 1. Since $f_0 > 0$, it follows that in a neighbourhood U_0 of the origin

$$f(x, x^{\cdot}) \geqslant m > 0$$

If the distance between f and g in the metric of $C_1(U_0)$ is at most $m/2$, then for $(x, x^{\cdot}) \in U_0$

$$g(x, x^{\cdot}) \geqslant m/2 > 0$$

Suppose that at time $t = t_0$ the point $(x_0, x_0^{\cdot})$ lies in U_0. Then the following inequalities hold for each of systems (1.1) and (2.1) at points of U_0

$$x^{\cdot\cdot} \geqslant m/2, \quad x^{\cdot} \geqslant x_0^{\cdot} + m(t - t_0)/2$$

so that after a time

$$\Delta t \leqslant \max_{U_0} |x_0^{\cdot}| \frac{4}{m}$$

the trajectories of both systems will leave U_0.

2. If $f_0 < 0$, there is a neighbourhood U_0 in which

$$0 < m \leqslant -f(x, x^{\cdot}) \leqslant M^2(1 - k^2)/2 \tag{2.2}$$

where m and M are constants. Consider the function

$$G(s, \upsilon) = \rho^3 + \alpha s \upsilon, \quad \rho^2 = \frac{(k^2 - 1)}{2f_0}\upsilon^2 + |s| \tag{2.3}$$

$$\alpha = M^{-1} \min\{\tfrac{1}{2}, 3k/(3 + k)\}$$

By (2.2), we have in U_0

$$|s| \leqslant \rho^2, \quad |\upsilon| \leqslant M\rho$$

The function G is positive definite, $\frac{1}{2}\rho^3 \leqslant G \leqslant \frac{3}{2}\rho^3$, and its derivative with respect to time is, by (1.4)

$$dG/dt = \tfrac{3}{2}\rho(\rho^2)^{\cdot} + \alpha\upsilon^2 R + \alpha R^{-1}[\rho^2 + (1-k^2)\upsilon^2/(2f_0)]f$$

$$(\rho^2)^{\cdot} = -2k|\upsilon| + |\upsilon|[k + \operatorname{sign}(s\upsilon)](1 - f/f_0)$$

Under our assumptions

$$-dG/dt = 3k\rho|\upsilon| - \tfrac{1}{2}\alpha\upsilon^2[3 - k\operatorname{sign}(s\upsilon)] - \alpha R^{-1}\rho^2 f +$$

$$+\tfrac{1}{2}(1 - f/f_0)[3\rho|\upsilon|(k + \operatorname{sign}(s\upsilon)) + \alpha(k^2-1)\upsilon^2 R^{-1}] \geqslant$$

$$\geqslant \alpha m\rho^2/(1+k) + o(\rho^2) = O(G^{2/3})$$

since by differentiability $f = f_0 = O(\rho)$.

Since the improper integral

$$J_0 = \int_0^{G_0} G^{-2/3} dG = 3G_0^{1/3}$$

is convergent, we conclude that G will vanish within a time of the order of ρ. When that happens the representative point will reach the origin and the system will reach equilibrium by condition (1.7). This case of system (1.1) describes quasi-plastic impact [7]. The phase portrait looks like Fig. 2(b), but unlike the example of Sec. 1, the trajectories will converge to the origin in a finite time.

3. If $f_0 = 0$, the type of singular point will depend on the values of the partial derivatives of f at the origin. System (1.4) becomes

$$s^{\cdot} = R\upsilon, \quad \upsilon^{\cdot} = R^{-1} f_1^{\circ} s + f_2^{\circ}\upsilon + \Phi(|s|, R\upsilon\operatorname{sign}s)R^{-1}\cdot\operatorname{sign}s \qquad (2.4)$$

$$f_1^{\circ} = \frac{\partial f}{\partial x}(0,0), \quad f_2^{\circ} = \frac{\partial f}{\partial x^{\cdot}}(0,0)$$

$$\Phi \in C_1, \quad \frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial x^{\cdot}} = 0 \quad \text{for} \quad x = x^{\cdot} = 0$$

The linear system obtained from (2.4) by putting $\Phi = 0$ has a singularity of one of the standard types at the origin: saddle-point, node, focus or centre (the last two were discussed in the example of Sec. 1).

For a function g close to f in the metric of C_r, the number g° is not necessarily equal to zero. Let us assume that $g^{\circ} > 0$. Then the origin is a "quasi-saddle-point" for system (2.1); in addition, the system may have other singular points near this one: for such points

$$\upsilon = 0, \quad g_1^{\circ} s + g^{\circ}\operatorname{sign}s + o(s) = 0, \quad g_1^{\circ} = \frac{\partial g}{\partial x}(0,0) \qquad (2.5)$$

If $g_1^{\circ} > 0$, the second equation of (2.5) has no roots, while if $g_1^{\circ} < 0$ it has two roots

$$s_{1,2} = \pm g^{\circ}/g_1^{\circ} + o(g^{\circ})$$

The discussion of the case $g^{\circ} < 0$ is similar.

Thus, arbitrarily small variations of the right-hand side of system (2.1) will cause both the number of singular points and their types to change.

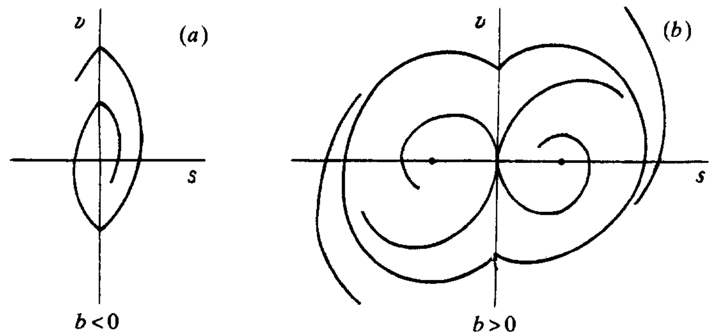

FIG. 3.

Example. In the oscillator (1.9), the origin represents contact of the impactor and the limiter. If $x_0 < 0$ the point is a quasi-focus, if $x_0 > 0$, a quasi-saddle-point. If $b < 0$, $\sigma < 1$, the point is a stable focus if $x_0 = 0$; variation of x_0 causes bifurcation, producing a quasi-saddle point, a pair of unstable foci and a stable limit cycle (Fig. 3; see also [8]).

3. THE STABILITY OF EQUILIBRIUM POSITIONS

Some results on the stability of equilibrium positions of a system of two equations with discontinuous right-hand sides were obtained in [9]. We will consider the stability of the equilibrium position of system (1.1) at the origin as a function of the value of f and its derivatives there.

If $f_0 > 0$, the origin is not an equilibrium position; if $f_0 < 0$, it follows from Theorem 1 that it is asymptotically stable. If $f_0 = 0$ the question of stability reduces to consideration of system (2.4).

Theorem 2. In each of the following cases

$$(1)\ f_1^\circ > 0$$

$$(2)\ f_1^\circ < 0, \quad D = (f_2^\circ)^2 + 4f_1^\circ > 0, \quad f_2^\circ > 0$$

$$(3)\ f_1^\circ < 0, \quad D < 0, \quad \sigma = \kappa \exp(\pi f_2^\circ / \sqrt{-D}) > 1;$$

the equilibrium position of system (2.4) at the origin is unstable for any non-linear part Φ; in the cases when

$$(4)\ f_1^\circ < 0, \quad D > 0, \quad f_2^\circ < 0$$

$$(5)\ f_1^\circ < 0, \quad D < 0, \quad \sigma < 1$$

it is asymptotically stable for any non-linear part Φ.

In all other cases the stability of the system depends on Φ.

Proof. To analyse cases 1, 2 and 4 (saddle-point, node), we note that if $\Phi = 0$ the linear system (2.4) has invariant lines whose slopes are the roots of the quadratic equation

$$z^2 - f_2^\circ z - f_1^\circ = 0, \quad z = Rv / s$$

If the non-linear terms are included, these lines determine the asymptotic directions for system (2.4): for a positive root z there is an asymptotic direction as $t \to -\infty$, for a negative root, as $t \to +\infty$. Since these directions are transverse to the impact line $s = 0$, the system, on approaching the singular point, will experience only a finite number of impacts against the limiter, and the presence of the unilateral constraint will not affect stability.

In cases 3 and 5 (a focus), we let $f_2{}^*$ denote a number such that σ, evaluated for specified $f_1{}^\circ$ and κ, equals unity. Then $f_2{}^\circ > f_2{}^*$ for an unstable focus and $f_2{}^\circ < f_2{}^*$ for a stable one. The trajectories of the auxiliary linear system

$$s^{\cdot} = R\upsilon, \quad \upsilon^{\cdot} = R^{-1} f_1^\circ s + f_2^* \upsilon \tag{3.1}$$

are closed (Fig. 2a), each of them intersecting the half-line $s = 0$, $\upsilon = 0$ at a single point $(0, \upsilon_0)$. Define $J(s, \upsilon) = \upsilon_0$. Then J is a first integral of system (3.1) (for the explicit form of this function, see [9]), i.e.

$$\frac{\partial J}{\partial s} R\upsilon + \frac{\partial J}{\partial \upsilon}(f_1^\circ R^{-1} s + f_2^* \upsilon) = 0 \tag{3.2}$$

The variables x, υ may be expressed in terms of J and t; by analogy with the example of the preceding section, the dependence on J is linear. By the formulae for differentiating implicit functions

$$1 = \frac{\partial J}{\partial s}\frac{\partial s}{\partial J} + \frac{\partial J}{\partial \upsilon}\frac{\partial \upsilon}{\partial J} = \frac{\partial J}{\partial s}\frac{s}{J} + \frac{\partial J}{\partial \upsilon}\frac{\upsilon}{J} \tag{3.3}$$

By (3.2) and (3.3)

$$\frac{\partial J}{\partial s} = -(f_2^* R\upsilon + f_1^\circ s)\frac{J}{Q}, \quad \frac{\partial J}{\partial \upsilon} = \frac{J}{Q} R^2 \upsilon \tag{3.4}$$

$$Q = R^2 \upsilon^2 - f_1^\circ s^2 - f_2^* R s \upsilon$$

where, in the cases considered, the quadratic form Q is positive definite. We note that J is a quantity of the first order in $r = (s^2 + \upsilon^2)^{1/2}$, while its partial derivatives (3.4) are bounded. We will construct a Lyapunov function: $L = J + \lambda s \upsilon$. This function is positive definite for all real λ, and by (3.4)

$$L^{\cdot} = J(f_2^\circ - f_2^*)\upsilon^2 R^2 / Q + \lambda f_1^\circ s^2 / R + \ldots \tag{3.5}$$

where the dots stand for smaller-order terms. The choice of λ is governed by the requirement that the function (3.5) be sign definite—with the same sign as the difference $f_2{}^\circ - f_2{}^*$. By Lyapunov's theorem, this implies the truth of our stability and instability assertions.

The derivative $L^{\cdot}$ is of the same order of magnitude as L. In cases 3 and 5, therefore, the motion towards the origin along a phase trajectory takes infinite time (unlike the quasi-focus case), and system (1.1) experiences an infinite number of collisions with the limiter.

4. THE STABILITY OF PERIODIC SOLUTIONS

A periodic solution of system (1.1) with impacts against the limiter corresponds to a closed trajectory Γ in the phase plane (s, υ), encircling the origin. Since equations (1.4) are invariant under the substitution $s, \upsilon \to -s, -\upsilon$, the curve symmetric to Γ about the origin will also be a phase trajectory. This limits the possible types of periodic motion when the right-hand side of Eq. (1.1) is not explicitly dependent on time, i.e. in autonomous systems. In addition, Γ cannot self-intersect at any point other than the origin, so that impacts take place at equally spaced times τ and at equal approach velocities $x^{\cdot -}$ (the motion of the representative point along Γ

takes a time τ).

If there are no other closed trajectories in the neighbourhood of Γ, it follows from the Poincaré–Bendixson theorem for systems with discontinuities that all the trajectories in this neighbourhood wind onto Γ as $t \to +\infty$ or as $t \to -\infty$.

The phase trajectories of a non-autonomous system may self-intersect, and the type of the periodic motion is characterized by two natural numbers: $N = \tau/T$ and the number K of impacts per period.

The different definitions of stability for periodic solutions may be formulated in the usual way. Note that in order to state these definitions directly for system (1.1) one must modify the concept of a neighbourhood for discontinuous trajectories [3], while on passing to point mappings [1] it is not always clear what type of stability is being discussed.

Orbital stability of an isolated periodic motion depends in an autonomous system on its asymptotic behaviour: if nearby trajectories wind onto Γ from inside and outside as $t \to +\infty$, the motion will be asymptotically stable; otherwise, it will be unstable. The behaviour of these curves may be ascertained by comparing the ordinates of their successive intersections with the half-line $s = 0$, $\upsilon > 0$. Thus, this half-line determines the Poincaré section in the investigation of orbital stability.

To analyse stability in Lyapunov's sense, one must compare the position of the representative points on Γ and on the perturbed trajectory at the same times. In that case the Poincaré section is defined by a plane $t = \text{const}$.

One example demonstrating the difference between the two types of stability is the motion of a heavy particle performing absolutely elastic collisions with a horizontal base; then $k = 0$ and system (1.4) is

$$s^{\cdot} = \upsilon, \quad \upsilon^{\cdot} = -\operatorname{sign} s$$

All the phase trajectories of this system are closed, but the periods of motion around different trajectories may be different. Thus each periodic motion is orbitally stable, but unstable in Lyapunov's sense.

An algorithm for investigating stability in the first approximation in systems with discontinuous right-hand sides was proposed in [10]. We shall apply that algorithm to periodic motion of system (1.4), on the assumption that the impactor never grazes the limiter (i.e. Γ never passes through the origin). Such a motion is represented in the (s, υ)-plane by a continuous curve $s_0(t)$, $\upsilon_0(t)$, which is either closed (if K is even) or passes at times $t = 0$ and $t = \tau$ through points symmetric about the origin. In the second case, one can consider instead of this motion a motion of type $(2N, 2K)$, whose trajectory in the (s, υ)-plane will be closed.

We may assume without loss of generality that $s_0(t_0)\upsilon_0(t_0) \neq 0$. At times for which $s_0(t)\upsilon_0(t) \neq 0$, the equations in variations $\xi = s - s_0$, $\eta = \upsilon - \upsilon_0$ for system (1.4) are

$$\xi^{\cdot} = R\eta, \quad \eta^{\cdot} = R^{-1} f_1 \xi + f_2 \eta \tag{4.1}$$

where the number R and partial derivatives of f are evaluated for the solution in question.

A fundamental matrix of solutions of system (4.1) satisfies the differential equation

$$\dot{X}(t) = \begin{Vmatrix} 0 & R \\ R^{-1} f_x & f_{x^{\cdot}} \end{Vmatrix} X(t) \tag{4.2}$$

At times when the trajectories (s_0, υ_0) intersect the coordinate axes, the matrix $X(t)$ changes abruptly: on passing from the first quadrant to the fourth, or from the third to the second

$$X^{+}(t) = \begin{Vmatrix} 1 & 0 \\ 0 & \kappa \end{Vmatrix} X^{-}(t) \tag{4.3}$$

(minus and plus superscripts denote the values of the matrix just before and after the intersection); in the reverse passages

$$X^{+}(t)=\begin{Vmatrix}1 & 0\\ 0 & \kappa^{-1}\end{Vmatrix}X^{-}(t) \tag{4.4}$$

and upon intersection with the axis $s=0$, corresponding to impacts in the original system (1.1)

$$X^{+}(t)=\begin{Vmatrix}\kappa & 0\\ \dfrac{1}{|\upsilon_0|(1+k)}\left[\dfrac{f^{-}}{1+k}+\dfrac{f^{+}}{1-k}\right] & 1\end{Vmatrix}X^{-}(t) \tag{4.5}$$

Stability depends on the roots ρ_1, ρ_2 of the characteristic equation

$$\det(X(\tau)-\rho E_2)=0,\quad X(0)=E_2 \tag{4.6}$$

The criterion for asymptotic stability is

$$|\rho_{1,2}|<1 \tag{4.7}$$

This inequality can be verified directly only after integrating Eqs (4.2)–(4.5), so it is important to develop easily verifiable conditions, even if only necessary or only sufficient. The next theorem establishes such a condition.

Theorem 3. If $x_0(t)$ is an asymptotically stable periodic motion of type (N, K), then

$$\kappa^{2K}\exp\left\{\int_0^{NT} f_{\dot x}(t,x_0(t),\dot x_0(t))\,dt\right\}<1 \tag{4.8}$$

Proof. Applying Liouville's theorem to system (4.2), we get

$$B^{\cdot}(t)=f_{\dot x}B(t),\quad B(t)=\det\{X(t)\},\quad B(0)=1 \tag{4.9}$$

To calculate the number $\beta(\tau)=\rho_1\rho_2$, we note that B changes abruptly across the coordinate axes, in such a way that $B^{+}=\kappa B^{-}$ for cases (4.3) and (4.5) and $B^{+}=B^{-}/\kappa$ for case (4.4). It is clear from easy geometric arguments that to each intersection of trajectories $(s_0,\ \upsilon_0)$ and the υ axis there corresponds an odd number of intersections of the s axis, and the number of type (4.3) is just one mote than the number of those of type (4.4). Solving the Cauchy problem for the linear equation (4.9) and noting that B is multiplied by κ^2 for each impact, we see that the left-hand side of Eq. (4.8) equals $B(\tau)$, so that the assertion of the theorem follows from the criterion (4.7).

Corollary. If the system is autonomous, condition (4.8) is necessary and sufficient for orbital asymptotic stability (as already pointed out above, in that case always $K=1$).

Indeed, in that case $\rho_1=1$ [11]; therefore, $B(\tau)$ is equal to ρ_2—the multiplier of the limit cycle Γ.

Specific problems are conveniently tackled using a simplified algorithm to construct the matrix $X(\tau)$. Let $X_1(\tau)$ denote a fundamental matrix of solutions of the linearized system (1.1). In the intervals of impact-free motion, the variation of this matrix is governed by the equation

$$X_1^{\cdot}(t)=\begin{Vmatrix}0 & 1\\ f_x & f_{\dot x}\end{Vmatrix}X_1(t),\quad X_1(t_0)=E \tag{4.10}$$

A comparison of systems (4.2) and (4.10) implies the following relationship between X and X_1

$$X(t)=\begin{Vmatrix}1 & 0\\ 0 & R^{-1}(t)\end{Vmatrix} X_1(t) \begin{Vmatrix}1 & 0\\ 0 & R(t_0)\end{Vmatrix} \operatorname{sign}(s_0(t)s_0(t_0)) \tag{4.11}$$

The matrix X_1 may be discontinuous only at impacts, when, by (4.5) and (4.11)

$$X_1^+=\begin{Vmatrix}-\kappa & 0\\ (f^+ +\kappa f^-)/x^{\cdot -} & -\kappa\end{Vmatrix} X_1^-$$
$$f^\pm = f(t,0,x^{\cdot\pm}) \tag{4.12}$$

Integrating system (4.10), (4.12), one can construct the monodromy matrix $X_1(\tau)$ which, by (4.11), is similar to $X(\tau)$.

Example. Let us consider forced oscillations of the oscillator (1.9)

$$x^{\cdot\cdot}+2bx^{\cdot}+a^2x=P(t), \quad P(t+T)\equiv P(t) \tag{4.13}$$

Suppose the system has a periodic motion $x(t)=p(t)$ of type $(N, 1)$ (i.e. one impact per period $\tau=NT$). Irrespective of the form of the right-hand side, the variational equations have the following general solution

$$x(t)-p(t)=e^{-bt}(C_1\cos\delta t+C_2\sin\delta t)$$

Applying formula (4.12) at the impact time $t=t^*$, we finally obtain the following expression for the coefficients of the characteristic equation

$$a_0=B(\tau)=\kappa^2 e^{-2bt} \tag{4.14}$$
$$a_1=\operatorname{tr}\{X_1(\tau)\}=e^{-b\tau}\left\{2\kappa\cos\delta\tau+\frac{(1+\kappa)P(t^*)\sin\delta\tau}{\delta p^{\cdot}(t^*-0)}\right\}$$

By Schur's theorem, the stability conditions (4.7) are now

$$|a_0|<1, \quad |a_1|<|1+a_0| \tag{4.15}$$

5. BIFURCATIONS IN VIBRO-IMPACT SYSTEMS

Let us assume now that the right-hand side of Eq. (1.1) depends on a parameter μ. The elements of the monodromy matrix for periodic motions with impacts are continuous functions of μ, as follows from the algorithm for constructing the matrix (4.10), (4.12). Hence the truth of inequality (4.7) guarantees that stable periodic motions will be preserved when μ is varied. Bifurcations will occur when one of the multipliers is on the unit circle; these bifurcations will be of one of the usual types—"saddle-node", period-doubling, and so on.

The so-called C-bifurcations are specific to impact systems [12]: if a periodic trajectory passes through the origin at some parameter value (the impactor grazes the limiter at some instant of time), the functions $\rho_{1,2}(\mu)$ may experience a discontinuity. Indeed, in that case the denominator in (4.12) will vanish—generally implying the appearance of infinitely large elements in the matrix $X_1(\tau)$. As a result of a C-bifurcation, whole families of periodic or

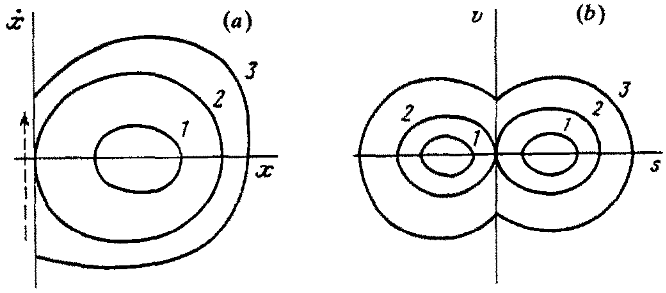

FIG. 4.

subperiodic motions may disappear or be created [13–15]. Feigin [13–15] believes that all these motions are unstable in the neighbourhood of the critical value of the parameter. However, we shall show presently that this statement is not entirely accurate.

Theorem 4. Let system (1.1) be autonomous and assume that f is a differentiable function of a parameter μ. For μ in an interval $(-\varepsilon, 0)$, where ε is a certain positive number, impact-free periodic motions exist whose multipliers are jointly bounded by a number λ less than unity

$$\rho_2(\mu) \leqslant \lambda < 1, \quad \mu \in (-\varepsilon, 0)$$

but for $\mu > 0$ there are no impact-free periodic motions (we recall that in the autonomous case $\rho_1(\mu) \equiv 1$). Then, for sufficiently small positive μ, there exist stable motions with one impact per period.

Proof. Picture the evolution of the periodic motions as in Fig. 4; μ is negative for curve 1, zero for curve 2 and positive for curve 3. The last-named trajectory is discontinuous in the $(x, x^{\cdot})$-plane plane but in the (s, v)-plane, however, it is closed, though the period is twice as large as in the original variables. Suppose we now draw positive and negative half-trajectories of system (1.1) through the origin and continue them until their first intersection with the x axis, at points x_μ^+, x_μ^-; at $\mu = 0$ the two points coincide. It follows from the existence of a periodic curve at $\mu = 0$ that, for sufficiently small μ, the function f is positive at the origin. In the domain $x > 0$, therefore, the phase trajectories are approximated by parabolas. Construct the successor function $x = g(\bar{x})$ [11] on the half-line $x > 0$. On impact, the representative point moves along the tangent to the parabolic trajectory through the origin; hence the distance from the point to the trajectory decreases by a factor of κ^2. Consequently, we obtain the following expression for the successor function

$$g(\bar{x}) = \begin{cases} x_\mu^+ + \rho_2(-0)(\bar{x} - x_\mu^-) + o(\mu), & \bar{x} \leqslant x_\mu^- \\ x_\mu^+ + \kappa^2 \rho_2(-0)(\bar{x} - x_\mu^-) + o(\mu), & \bar{x} > x_\mu^- \end{cases} \tag{5.1}$$

When $\mu < 0$, orbital asymptotic stability implies $x_\mu^+ < x_\mu^-$; if $\mu > 0$ the non-existence of impact-free periodic motions implies $x_\mu^+ > x_\mu^-$. Since $\rho_2(-0) < 1$, formula (5.1) describes a two-sided bifurcation: when $\mu < 0$ the map g has a fixed point

$$x_1^* = x_\mu^- + (x_\mu^+ - x_\mu^-)/(1 - \rho_2(-0)) < x_\mu^-$$

and when $\mu > 0$ the fixed point is

$$x_2^* = x_\mu^- + (x_\mu^+ - x_\mu^-)/(1 - \kappa^2 \rho_2(-0)) > x_\mu^-$$

Since $g'(x_1^*) = \rho_2(-0)$, $g'(x_2^*) = \kappa^2 \rho_2(-0)$, both these points are stable.

Remark. If $\rho_2(-0)\in(1,\ \kappa^{-2})$, the bifurcation is one-sided: both fixed points $x_{1,2}^*$ exist for $\mu<0$ (thus, curve 3 in Fig. 4, like curve 1, is drawn for $\mu<0$). Since $g'(x_1^*)>1,\ g'(x_2^*)<1$, the impact-free motions are unstable, while those with impacts are stable.

In non-autonomous systems, the trajectories may self-intersect in the phase plane, so that a C-bifurcation may cause changes in the period or in the number of impacts per period.

Let us assume that for parameter values $\mu\in(-\varepsilon,\ 0]$ a family $x_\mu(t)$ of τ-periodic motions of a non-autonomous system exists, exactly one of which involves grazing incidence of the impactor and the limiter

$$x_0(t_0)=\dot{x}_0(t_0)=0,\quad f(t_0,0,0)>0 \tag{5.2}$$

Form the monodromy matrix $X_1(\mu,\ t_0+\tau)$, by solving system (4.10) with condition (4.12). Denote its elements by $x_{ij}(\mu)$ $(i,\ j=1,\ 2)$.

Theorem 5. A necessary condition for the existence of a stable τ-periodic motion when $\mu>0$, with one extra impact per period compared with motions with $\mu<0$, is that

$$x_{12}(-0)=0 \tag{5.3}$$

Proof. The existence of an additional impact in the neighbourhood of $t=t_0$ implies that the monodromy matrix has a jump, in accordance with (4.12). The determinant is then multiplied by κ^2, while the trace is increased by a quantity proportional to $x_{12}(\mu)/x^{*-}$. Since $x^{*-}\to 0$ as $\mu\to 0$, we see that condition (5.3) is indeed necessary for the truth of the second inequality in (4.15).

Example. For the oscillator (4.13), if $P(t_0,\ \mu)\neq 0$, the necessary condition (5.3) becomes

$$\sin\delta\tau=0 \tag{5.4}$$

Let us construct periodic motions when the applied load is described by the formula

$$P(t,\mu)=(1-\mu)a^2+2b\omega\sin\omega t+(\omega^2-a^2)\cos\omega t$$

System (4.13) has a particular solution

$$p_\mu(t)=1-\mu-\cos\omega t,\quad T=2\pi/\omega \tag{5.5}$$

If $\mu<0$, this formula describes an impact-free periodic motion. If $\mu=0$ the impactor grazes the limiter at times $t_0=0,\ \pm T,\ \pm 2T$, and so on. But if $\mu>0$, there are no impact-free motions.

A periodic motion of type (1.1) is described in intervals between impacts by the formula

$$x_\mu(t)=p_\mu(t)+e^{-bt}(C_1\cos\delta t+C_2\sin\delta t)$$

If the impacts take place at times $t^*+mT,\ m\in Z$, the T-periodicity conditions are

$$x_\mu(t^*)=x_\mu(t^*+T)=0,\quad \dot{x}_\mu(t^*)=-\kappa\dot{x}_\mu(t^*+T)\geqslant 0 \tag{5.6}$$

By (5.4), conditions (5.6) may be considerably simplified

$$\delta=\tfrac{1}{2}n\omega,\quad n\in Z,\quad t^*=-\omega^{-1}\arccos(1-\mu) \tag{5.7}$$

$$C_1\cos\delta t^*+C_2\sin\delta t^*=0,\quad C_2=\frac{-\omega(1+\kappa)\sin\omega t^*\cos\delta t^* e^{b(t^*+T)}}{\delta(e^{bT}+\kappa(-1)^n)}$$

A direct check will show that the geometric conditions for the solution of (5.7) to exist are satisfied, that is, the function $x_\mu(t)$ is positive in the interval (t^*, t^*+T).

In formulae (4.14) we obtain

$$a_0 = \kappa^2 e^{-2b\tau}, \quad a_1 = \pm 2\kappa e^{-b\tau}$$

Consequently, a C-bifurcation will produce a stable periodic motion with impacts.

By analogy with Theorem 5, one can formulate a necessary condition for a C-bifurcation to produce stable motions of type $(N, K+1)$. This condition is

$$x_{12}^{(N)}(-0) = 0 \tag{5.8}$$

where $x_{ij}^{(N)}(\mu)$ are the elements of the Nth power of the monodromy matrix $X_1(t_0+T)$.

In our example, condition (4.8) looks as follows:

$$\sin(N\delta T) = 0 \tag{5.9}$$

Remark. Equations (5.4) and (5.9) actually state that the natural frequency of the damped oscillations of the oscillator and the frequency of the periodic load are commensurable. In most situations in mechanics, resonances constitute a destabilizing factor. In this case, however, resonances cause stability to be preserved under C-bifurcation. As shown in [16], stable periodic motions of system (4.13) may also appear near a resonance: if the left-hand side of Eq. (5.4) is a negative number close to zero, C-bifurcations are preceded by a "saddle-node" bifurcation, which gives rise to a pair of motions with non-degenerate impacts. One of these motions degenerates and annihilates together with the impact-free motion at $\mu = 0$; the stable motion with impacts, however, is preserved when $\mu > 0$.

In non-autonomous systems, there is yet another case in which stability is preserved under a C-bifurcation. For it to occur, the following conditions must hold when $\mu = 0$

$$f(t_0,0,0) = \frac{d}{dt} f(t_0,0,0) = 0, \quad \frac{d^2}{dt^2} f(t_0,0,0) > 0$$

The scenario is depicted approximately in Fig. 5: in a trajectory of the third type, the number of impacts per period is twice as large as for trajectory 1. It is understood here that at the instant trajectory 3 intersects the υ axis, we have $f = 0$, so that, by (4.12), the "extra" impacts cause the multipliers to be multiplied by κ^2.

6. GENERALIZATION OF THE METHOD OF CONTINUOUS REPRESENTATION

The method of continuous representation of discontinuous oscillations, as described in Sec. 1, may also be used to analyse multi-dimensional systems with one impact pair, including cases when the coefficient of restitution depends on the initial impact conditions or when dry friction must be taken into account.

Let $\mathbf{y} = (x, x^\cdot, z_1, \ldots, x_n)$ be the vector of phase variables, $x \geq 0$. The motion in the domain $x > 0$ is described by the equations

$$\mathbf{y}^\cdot = \mathbf{F}(t, \mathbf{y}) \tag{6.1}$$

In the $x = 0$ plane the variation of $x^\cdot$ is governed by a rule analogous to (1.2), while the remaining equations of system (6.1) remain unchanged. In particular, mechanical systems with an ideal unilateral constraint may be reduced to the form of (6.1) [5]: in that case z will be Lagrangian coordinates and the conjugate momenta; the equations of motion will be written in Routh form.

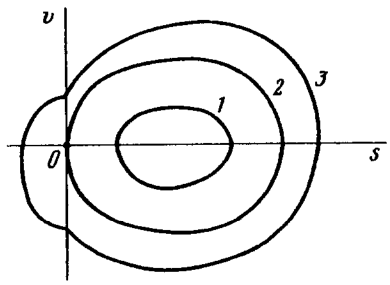

FIG. 5.

We make the substitution (1.3), leaving z unchanged. The continuous representation of system (6.1) will be

$$s^{\cdot} = R\upsilon, \quad \upsilon^{\cdot} = R^{-1}F_2 \operatorname{sign} s, \quad z_j^{\cdot} = F_{j+2} \quad (j = 1,\dots,n) \tag{6.2}$$

where the right-hand sides are discontinuous when $s\upsilon = 0$.

In the more-general case, when the impact interactions are described by the formulae

$$x^{\cdot +} = -x^{\cdot -} H_0(y^-), \quad 0 < H_0 < 1 \tag{6.3}$$
$$z_j^+ = z_j^- + x^{\cdot -} H_j(y^-) \quad (j = 1,\dots,n)$$

the auxiliary phase variables $s, \upsilon, w_1, \dots, w_n$ may be defined as follows:

$$x = |s|, \quad x^{\cdot} = \upsilon \operatorname{sign} s (1 - k^* \operatorname{sign}(s\upsilon)) \tag{6.4}$$
$$z_j = w_j + x^{\cdot}\alpha_j(|\upsilon|, w_1,\dots,w_n), \quad k^* = (1 - H_0)/(1 + H_0)$$

The functions α_j are chosen so that the variables υ, w_j remain unchanged under the impact transformations (6.3). To that end, they must satisfy the system of equations

$$\alpha_j = -H_j/(1 + H_0)$$

where we have put $x^{\cdot -} = -|\upsilon|(1 + k^*)$.

Differentiating Eqs (6.4) on the assumption that $s\upsilon \neq 0$, one obtains a continuous representation of system (6.1) with impacts (6.3).

Example. A material particle falls at an angle onto a rough horizontal plane. The equations of the impact are

$$x^{\cdot +} = -\kappa x^{\cdot -}, \quad z^{\cdot +} = z^{\cdot -} + \mu(1 + \kappa)x^{\cdot -} \tag{6.5}$$

where x, z are the Cartesian coordinates of the particle (x is the height above the base plane) and μ is the coefficient of Coulomb friction.

The second equation of (6.5) is true if the value of $z^{\cdot +}$ calculated using it is positive, and the same holds for $z^{\cdot -}$; otherwise one must assume that $z^{\cdot +} = 0$, i.e. the sliding of the particle terminates. Formulae (6.5) reduce to (6.3) if we put $H_0 = \kappa$, $H_1 = \mu(1 + \kappa)$. Consequently, the auxiliary variable w may be defined as $w = \mu x^{\cdot} + z^{\cdot}$.

In the inter-impact intervals $x^{\cdot\cdot} = -g$, $z^{\cdot\cdot} = 0$. Therefore, the continuous representation of the system is

$$s^{\cdot} = R\upsilon, \quad \upsilon^{\cdot} = -gR^{-1} \operatorname{sign} s \tag{6.6}$$

$$z^{\cdot} = w - \mu R\upsilon \text{sign}s, \quad w^{\cdot} = -\mu g \quad (w \geqslant 0)$$

where g is the acceleration due to gravity, and the mass of the particle is unity.

System (6.6) is easily integrated. The first two equations describe the motion of a particle falling perpendicularly onto the base, in the form of an infinite impact process that is damped out in finite time [7]. The fourth equation is formally the same as that of the motion of a body sliding along the rough surface (at a velocity w); if the initial value of the velocity $z^{\cdot}$ is sufficiently large, this equation will remain valid after the jumps stop.

Using the continuous representation, one can establish various results for multi-dimensional systems, relating to the stability of equilibrium positions and periodic motions with impacts. Thus, let us assume that $F_3 = F_4 = \ . \ . \ . \ = F_{n+2} = 0$, $F_2 < 0$ for $y = 0$. Then the origin is an equilibrium position of system (6.2). Consider the auxiliary system of equations

$$z_j^{\cdot} = F_{j+2}\big|_{s=\upsilon=0} \quad (j = 1, 2, \ldots, n) \tag{6.7}$$

The right-hand sides of Eqs (6.7) are continuously differentiable in the neighbourhood of the singular point $z = 0$; the type of singularity may be determined in the usual way.

Further conclusions as to stability will rely on the following theorem.

Theorem 6. If for all $t \geqslant t_0$

$$-M \leqslant F_2(t, 0) \leqslant -m < 0$$

then the trivial solution of system (6.1) is stable (asymptotically stable) if and only if the trivial solution of system (6.7) is stable (asymptotically stable).

Proof. The assertion will obviously be true if we can show that, for all initial data in some neighbourhood U_0 of the origin, the phase trajectory of system (6.1) will reach the plane $x = x^{\cdot} = 0$ in a finite time and remain there at least until it leaves U_0. This property follows from an analysis of functions (2.3) satisfying conditions (3.2). Although the derivative of the function $G(s, \upsilon)$ for system (6.2) may be explicitly time-dependent, it will satisfy the same estimate. Repeating the arguments used to prove Theorem 1, one sees that G will vanish in a finite time, i.e. the trajectory will reach the plane $x = x^{\cdot} = 0$.

Example. The motion of a heavy particle on a sinusoidally vibrating base is described by the equation

$$d^2x / d\varphi^2 = I\sin\varphi - 1, \quad x \geqslant 0 \tag{6.8}$$

where φ is the phase of the oscillations, I is their intensity and x is a non-dimensional quantity proportional to the height of the particle above the moving base.

By Theorem 6, if $I < 1$ the origin is an asymptotically stable equilibrium position (the particle moves without breaking away from the base). This does not mean that there are no periodic motions with jumps [1]; for example, if

$$I > \pi k N, \quad N \in Z$$

there are periodic motions of period $\tau = 2\pi N$ with impacts in phase φ_0, where

$$I\cos\varphi_0 = \pi k N, \quad x^{\cdot -} = -(1 + k)\pi N \tag{6.9}$$

Let us analyse the stability of this periodic motion. The coefficients of the characteristic equation may be found by using formulae (4.10) and (4.12)

$$a_0 = \kappa^2, \quad a_1 = (1 + \kappa)^2 I\sin\varphi_0 - 1 - \kappa^2$$

so that conditions (4.7) become

$$0 < \operatorname{tg}\varphi_0 < \frac{2}{\pi N}\frac{1+\kappa^2}{1-\kappa^2}$$

agreeing with the results obtained in [1].

We will find the sufficient conditions for the stability of the origin of coordinates of system (6.8) as a whole. Consider the function

$$V = \tfrac{1}{2}(1-k^2)I\upsilon^2 + |s| \tag{6.10}$$

The derivative of the function (6.10) when $s\upsilon \neq 0$ is

$$dV/d\varphi = -|\upsilon|[2k - I\sin\varphi(k + \operatorname{sign}(s\upsilon))] \tag{6.11}$$

Consequently, if

$$I < 2k/(1+k) \tag{6.12}$$

the right-hand side of (6.11) will be negative, and the function (6.10) will satisfy the conditions of the Barbashin–Krasovskii theorem. Thus, condition (6.12) is sufficient for global stability.

REFERENCES

1. KOBRINSKII A. A. and KOBRINSKII A. Ye., *Two-dimensional Vibro-impact Systems.* Nauka, Moscow, 1981.
2. ZHURAVLEV V. F. and KLIMOV D. M., *Applied Methods in the Theory of Oscillations.* Nauka, Moscow, 1988.
3. GAJEWSKI K. and RADZISZEWSKI B., On the stability of impact systems. *Bul. Pol. Acad. Sci. Tech. Sci.* **35,** 3–4, 183–188, 1987.
4. FILIPPOV A. F., *Differential Equations with Discontinuous Right-hand Side.* Nauka, Moscow, 1985.
5. ZHURAVLEV V. F., Equations of motion of mechanical systems with ideal unilateral constraints. *Prikl. Mat. Mekh.* **42,** 781–788, 1978.
6. BAUTIN N. N. and LEONTOVICH Ye. A., *Methods and Rules for the Qualitative Study of Dynamical Systems in a Plane.* Nauka, Moscow, 1990.
7. NAGAYEV R. F., *Mechanical Processes with Recurring Damped Collisions.* Nauka, Moscow, 1985.
8. GUBAR' N. A., On bifurcations in the neighbourhood of a "sewn focus". *Prikl. Mat. Mekh.* **35,** 937–941, 1971.
9. SOLNTSEV Yu. K., The stability in Lyapunov's sense of equilibrium positions of a system of two differential equations in the case of discontinuous right-hand sides. *Uch. Zapl Moskov. Gos. Univ., Ser. Mat.* 148, 144–180, 1951.
10. AIZERMAN M. A. and GANTMAKHER F. R., The stability of periodic motions. *Prikl. Mat. Mekh.* **22,** 750–758, 1958.
11. ANDRONOV A. A., VITT A. A. and KHAIKIN S. E., *The Theory of Oscillations.* Nauka, Moscow, 1981.
12. BRUSIN V. A., NEIMARK Yu. I. and FEIGIN M. I., Some cases of the dependence of the periodic motions of a relay system on the parameters. *Izv. Vuzov. Radiofizika* **6,** 4, 785–800, 1963.
13. FEIGIN M. I., Period-doubling of oscillations in C-bifurcations in piecewise-continuous systems. *Prikl. Mat. Mekh.* **34,** 5, 861–869, 1970.
14. FEIGIN M. I., The creation of a family of subharmonic modes in a piecewise-continuous system. *Prikl. Mat. Mekh.* **38,** 5, 810–818, 1974.
15. FEIGIN M. I., The behaviour of dynamical systems near the boundaries of the domain of existence of periodic motions. *Prikl. Mat. Mekh.* **41,** 4, 628–636, 1977.
16. IVANOV A. P., Stabilization of an impact oscillator near grazing incidence owing to resonance. *J. Sound Vibr.* **162,** 3, 515–518, 1993.

Translated by D.L.